Endogenous Time Preference, Consumption Externalities, and Trade: Multiple Steady States and Indeterminacy

Yukio KARASAWA-OHTASHIRO*

Faculty of Economics

Nanzan University

Akihiko YANASE†

GSICS

Tohoku University

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Abstract

This paper presents a two-sector dynamic general equilibrium model in an autarkic economy and a small open economy, and considers the role of endogenous impatience and consumption externalities in a neoclassical growth model. In the case of socially increasing marginal impatience, there exists a unique and saddle-point stable steady state. By constrast, in the case of socially decreasing marginal impatience, there may exist multiple steady states and the dynamic equilibrium around the steady state with incomplete specialization may exhibit indeterminacy. The occurrence of the indeterminacy result requires a large degree of increasing time preference in average consumption and a socially decreasing marginal impatience which means that the slope of the supply curve for capital is negative in the long-run. In addition, the long-run effects of a terms-of-trade deterioration on the economy's comparative advantage and its relation to indeterminacy are examined.

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*Corresponding author: Faculty of Economics, Nanzan University, 18 Yamazato-cho, Showa-ku, Nagoya 466-8673, Japan. Phone: +81-52-832-3111; Fax: +81-52-835-1444; E-mail: yukioko@nanzan-u.ac.jp.

†Graduate School of International Cultural Studies, Tohoku University, 41 Kawauchi, Aoba-ku, Sendai 980-8576 Japan. Phone: +81-22-795-7556; Fax: +81-22-795-7583; E-mail: yanase@intcul.tohoku.ac.jp.

1. Introduction

This paper presents a two-sector dynamic small country model of capital accumulation and international trade with endogenous impatience and consumption externalities. Attention is focused on how the consumption externalities affect the long-run specialization patterns and whether these externalities are a source of indeterminacy (i.e., continuum of dynamic equilibrium paths converging to a common steady state).

Since the 1960s, the standard Heckscher-Ohlin-Samuelson (HOS) model of international trade has been extended to dynamic models by introducing capital accumulation. A prototype model was developed by Oniki and Uzawa (1965), who assume that a pure consumption good and a pure investment good are produced by employing labor and capital. The Oniki-Uzawa model, which is examined as a growth model in a neoclassical growth framework with constant savings rate, has been extended to ones in the optimal growth framework with endogenous savings by Stiglitz (1970), Manning (1981), Baxter (1992), Chen (1992), Nishimura and Shimomura (2002a, 2002b), and Karasawa (2007). Most of these optimal growth models, however, have the following unsatisfactory features: (*i*) in the steady state, a small open economy is likely to specialize completely in one good; (*ii*) the long-run equilibria with diversified production are not unique; and (*iii*) there is no transition dynamics for the long-run equilibria with diversified production.[1] This is because in the steady state, the rate of return to capital must be equal to the rate of time preference, which can be attained by chance for a small open economy if the economy is incompletely specialized (wherein the factor prices are completely dependent on the world commodity prices) and if the rate of time preference is an exogenous variable.

Nishimura and Shimomura (2002b) and Karasawa (2007) are a few exceptional studies that show that a small open economy incompletely specializes in both goods by assuming an endogenous time preference in which a representative household's discount rate is dependent on her/his consumption.[2] The dynamic model with endogenous time preference was introduced by Uzawa (1968), and has been extended and clarified by Epstein (1987) and Obstfeld (1990). These dynamic models have the properties "increasing marginal impatience," i.e., a positive relation-

[1] Baxter (1992, p.714) argued for these properties that "the predictions of this model regarding patterns of specialization and trade are markedly different from those of HOS model but are very much in the spirit of the traditional Ricardian model."

[2] Diversified production in the steady state is also made possible in Chen (1992), who introduces an endogenous labor supply. In this case, the factor prices are linked to economic variables other than commodity prices.

ship between discount rate and consumption. Although the assumption of increasing marginal impatience may be counterintuitive, this assumption has been standard in dynamic models with endogenous time preference, possibly because of its effectiveness in ensuring stability of the steady state.[3] However, Das (2003) and Chang (2009) proved that contrary to the general belief, a negative relationship between the discount rate and consumption does not necessarily result in the instability of the dynamic system in the closed one-sector economy.[4] Nishimura and Shimomura (2002b), which assumes externalities on the production technologies, and Karasawa (2007), which does not assume such externalities, examine both the Uzawa–Epstein increasing marginal impatience and the Das–Chang decreasing marginal impatience in models of a small open economy.

In this paper, we assume that households' discount rate as well as her/his felicity not only depends on their individual consumption, as assumed in Nishimura and Shimomura (2002b) and Karasawa (2007), but also on the economy's average consumption. This assumption has the following implications. First, the dependence of discount rate on average consumption implies consumption externalities that influence individual households' lifetime utility. There are two concepts regarding the manner in which the consumption externalities affect individual's utility: one is "jealous" or "admiring," as defined by Dupor and Liu (2003); and the other is "keeping up with the Joneses" or "running away from the Joneses," as defined by Galí (1994). These definitions of consumption externalities are in terms of the felicity function, but, as discussed latter, can be easily extended to lifetime utility. Second, as recent studies demonstrate, the consumption externalities can be a source of indeterminacy. Meng (2006) presents a dynamic model in which individual households' time preference depends on the economy's average levels of consumption and income, and shows that local indeterminacy can arise if the individual rate of time preference increases with average consumption and decreases with average income. Chen and Hsu (2007) introduce consumption externality into households' felicity function in a one-sector optimal growth model, and show that if the felicity is increasing in consumption externality (i.e., if individuals feel admiration) and if individual time preference

[3]Assuming a small open economy, dynamic trade models with the Uzawa–Epstein increasing marginal impatience are analyzed by Obstfeld (1982), Shi and Epstein (1993), Shi (1994), and Bian and Meng (2004). Recently, Chen et al. (2008) develop a dynamic two-country HOS model with the Uzawa–Epstein endogenous time preference and derive long-run trade-pattern propositions.

[4]The idea of decreasing marginal impatience dates back to Fisher (1930).

exhibits decreasing marginal impatience, indeterminacy can arise. Alonso-Carrera et al. (2008) show that consumption externalities are a source of equilibrium indeterminacy in a growth model with endogenous labor supply.

During the past decade, dynamic general equilibrium models that display indeterminacy have drawn increasing attention. Benhabib and Farmer (1994) is known as a pioneering work, followed by, e.g., Benhabib and Farmer (1999), Benhabib et al. (2000), and Mino (2001). There are also dynamic open-economy models that demonstrate the possibility of indeterminacy: a two-country trade model is examined by Nishimura and Shimomura (2002a), a small-country trade model by Nishimura and Shimomura (2002b) and Bian and Meng (2004), and a small-country model with perfect international capital market by Meng and Velasco (2004). The source of indeterminacy in these studies is intra- or inter-sectoral production externalities. On the other hand, consumption externalities have drawn little attention in the studies on indeterminacy. As mentioned above, Meng (2006), Chen and Hsu (2007), and Alonso-Carrera et al. (2008) are the exceptions, which examine closed economy models. Therefore, our study will be the first to analyze an open economy model displaying indeterminacy of equilibrium caused by consumption externalities even in the absence of production externalities.

The main results of this paper are summarized as follows. In the case of the socially increasing marginal impatience (i.e., the net effect of average and individual consumption on the discount rates is positive), the steady state exhibits uniqueness and saddle-point stability, irrespective of whether the economy is completely or incompletely specialized. In the case of the socially decreasing marginal impatience, by contrast, we may have multiple steady states: one with incomplete specialization and the rest with diversified production. Moreover, the long-run equilibrium with diversified production can be locally indeterminate if the following two conditions occur at the same time: individual's time preference is positively related to the average consumption, indicating that the household's expectation of a higher average consumption leads to further discounting of the future utility, and there is a socially decreasing marginal impatience, indicating that the long-run supply curve of capital is downward-sloping. These results are in contrast with Nishimura and Shimomura (2002b), who assume externalities on the production side only and show that indeterminacy is possible for the case of increasing marginal impatience while decreasing marginal impatience leads to saddle-point stability.

We also examine the effects of terms-of-trade deterioration on the steady-state solutions of the economy. We observe that the comparative static results concerning the terms-of-trade change are dependent on whether the marginal impatience is socially increasing or decreasing. This suggests that the comparative static results can be reversed depending on whether the dynamic equilibrium exhibits saddle-point stability or indeterminacy.

The rest of this paper is organized as follows. Sections 2 and 3 present the model and derives the optimality conditions. In section 4, we characterize the dynamic general equilibrium of the small open economy system in this model. In sections 5 and 6, we derive the properties of the dynamic system. In section 7, we consider the effects of terms-of-trade deterioration on the long-run equilibrium for the case of incomplete specialization. Section 8 concludes the paper.

2. Firms

We consider a small open economy in which two goods, good 1 (pure consumption good) and good 2 (pure investment good, which is assumed to be numeraire) are produced by employing capital and labor. Following Oniki and Uzawa (1965), we assume that the economy freely trades the two goods in the international market, while the capital and labor are internationally immobile.

Technology in each sector is specified by a production function, which is homogenous of degree one in both factors and is stationary over time. The output per worker in sector i must satisfy $y_i(t) \leq l_i(t) f_i(k_i(t))$, where $k_i(t)$ is the capital-labor ratio and $l_i(t) \in [0,1]$ is the proportion of the labor employed, respectively, in sector i at time $t \in [0,\infty)$, $i = 1,2$. We impose the following assumptions on the production function $f_i(k_i)$:

Assumption 1 (the production function)**.** *The production function $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ is twice continuously differentiable, with*

1. $f_i'(k_i(t)) > 0$ *and* $f_i''(k_i(t)) < 0$ $\forall k_i(t) > 0,\ i = 1,2,$
2. $\lim_{k_i(t)\to 0} f_i'(k_i(t)) = \infty,\ \lim_{k_i(t)\to\infty} f_i'(k_i(t)) = 0,$ *and* $f_i(0) = 0,\ i = 1,2.$

With full mobility of factors across sectors, the resource constraints require that $l_1(t)k_1(t) + l_2(t)k_2(t) \leq k(t)$ and $l_1(t) + l_2(t) \leq 1$, where $k(t) \in [0,\infty)$ is the aggregate capital-labor ratio.

Then, the per capita GDP function can be defined as

$$g\left(k\left(t\right),p\left(t\right)\right)\equiv\max_{\left\{y_i(t),k_i(t),l_i(t)\right\}}\left\{\mathbf{p}\left(t\right)\cdot\mathbf{y}\left(t\right):\ y_i\left(t\right)\leq l_i\left(t\right)f\left(k_i\left(t\right)\right),\ \sum_i l_i\left(t\right)k_i\left(t\right)\leq k\left(t\right),\right.$$
$$\left.\sum_i l_i\left(t\right)\leq 1,\ y_i\left(t\right)\geq 0,\ k_i\left(t\right)\geq 0,\ l_i\left(t\right)\geq 0,\ i=1,2\right\},\tag{P}$$

where $p\left(t\right)$ is the price of good 1 in terms of good 2, $\mathbf{y}\left(t\right)\equiv\left(y_1\left(t\right),y_2\left(t\right)\right)$, and $\mathbf{p}\left(t\right)\equiv\left(p\left(t\right),1\right)$. In optimality, $y_i\left(t\right)=l_i\left(t\right)f_i\left(k_i\left(t\right)\right)$, $l_1\left(t\right)+l_2\left(t\right)=1$, and $l_1\left(t\right)k_1\left(t\right)+l_2\left(t\right)k_2\left(t\right)=k\left(t\right)$ hold.

From Assumption 1, the GDP function has the properties

$$\lim_{k(t)\to 0}g_k\left(k\left(t\right),p\left(t\right)\right)=\infty,\quad \lim_{k(t)\to\infty}g_k\left(k\left(t\right),p\left(t\right)\right)=0,\quad g\left(0,p\left(t\right)\right)=0,$$

and is concave in $k\left(t\right)$ and convex in $p\left(t\right)$ as in Figure 1, which depicts the case where $\left(k\left(t\right),p\left(t\right)\right)=\left(k_0,p_0\right)$.[5] This figure also shows that it is the upper envelope of $p\left(t\right)f_1\left(k_1\left(t\right)\right)$ and $f_2\left(k_2\left(t\right)\right)$.

[Figure 1 about here.]

If both goods are produced, then it follows from the first-order conditions for profit maximization that the value of the marginal product in sector 1 should be equal to that in sector 2:

$$\begin{aligned}&p\left(t\right)f_1'\left(k_1\left(t\right)\right)=f_2'\left(k_2\left(t\right)\right),\\&p\left(t\right)\left[f_1\left(k_1\left(t\right)\right)-k_1\left(t\right)f_1'\left(k_1\left(t\right)\right)\right]=f_2\left(k_2\left(t\right)\right)-k_2\left(t\right)f_2'\left(k_2\left(t\right)\right).\end{aligned}$$

From these equations, the capital-labor ratio $k_i(t)$ can be rewritten as a function of $p(t)$: $k_i(p(t))$.

Assumption 2 (factor intensity ranking)**.** *Investment good is more capital-intensive and there are no factor intensity reversals, that is,* $k_2\left(p\left(t\right)\right)>k_1\left(p\left(t\right)\right)$, $\forall p\left(t\right)\in\left(0,\infty\right)$.[6]

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[5] Appendix A provides a detailed description of Figure 1. See also Deardorff (1974).

[6] We can solve the case consumption-good sector is more capital-intensive, that is, $k_2\left(t\right)<k_1\left(t\right)$, $\forall p\left(t\right)\in\left(0,\infty\right)$, in the similar way.

Let us define $p_i(k(t)) \equiv k_i^{-1}(k(t))$, $\forall k(t) \in [0,\infty)$, $i = 1,2$. Then, the GDP function has the following properties:[7]

$$\begin{cases} g_{kk}(k(t),p(t)) \leq 0, & \text{with equality when } p(t) \in (p_2(k(t)), p_1(k(t))), \\ g_p(k(t),p(t)) \geq 0, & \text{with equality when } p(t) \in (0, p_2(k(t))], \\ g_{pp}(k(t),p(t)) \geq 0, & \text{with inequality when } p(t) \in (p_2(k(t)), p_1(k(t))). \end{cases}$$

Moreover, Lemmas 1–3 in Appendix B provide the properties of $g_{kp}(k(t),p(t))$.

By using the envelope theorem, we have the following properties for the GDP function:

$$y_1(t) = g_p(k(t),p(t)) \equiv y_1(k(t),p(t)), \tag{1a}$$

$$y_2(t) = g(k(t),p(t)) - p(t)\,g_p(k(t),p(t)) \equiv y_2(k(t),p(t)), \tag{1b}$$

$$r(t) = g_k(k(t),p(t)) \equiv r(k(t),p(t)), \tag{1c}$$

$$w(t) = g(k(t),p(t)) - k(t)\,g_k(k(t),p(t)) \equiv w(k(t),p(t)), \tag{1d}$$

$$p(t)\,y_1(t) + y_2(t) = r(t)\,k(t) + w(t) = g(k(t),p(t)), \tag{1e}$$

where $r(t)$ is the rental rate of capital and $w(t)$ is the wage rate, respectively, in terms of good 2. Note that Eq. (1c) defines the demand function for capital in the production sector, and has slope $g_{kk}(k(t),p(t)) \leq 0$ in the $(k(t),r(t))$ plane; this is shown in Figure 2.[8]

[Figure 2 about here.]

3. Households

Consider an infinitely lived consumer who maximizes the lifetime utility that depends on the time profile of private consumption $c(t)$ and average consumption $C(t)$, i.e., $E \equiv \{c(t), C(t)\}_{t=0}^{\infty}$. It can be represented by a partitioned vector $(E_T, {}_TE)$ for all $T \in (0,\infty)$, where $E_T \equiv \{c(t), C(t)\}_{t=0}^{T}$ and ${}_TE \equiv \{c(t), C(t)\}_{t=T}^{\infty}$ are the programs up to time T and after T, respectively. The intertemporal utility function is defined over such programs in the following

[7] See, e.g., Wong (1995) for details.
[8] See also Stiglitz (1970).

manner:

$$U(E) \equiv \int_0^\infty u(c(t), C(t)) \exp\left[-\int_0^t \{\delta(c(t), C(t)) - n\} d\tau\right] dt, \tag{2}$$

where $u(c(t), C(t))$ is the "felicity" function, $\delta(c(t), C(t))$ is the subjective discount function, and $n \geq 0$ is the population growth rate of the economy.

Assumption 3 (the felicity function)**.** *The felicity function* $u : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_{++}$ *is real valued, bounded above, twice continuously differentiable in* $c(t)$ *with* $u_c(c(t), C(t)) > 0$ *and* $u_{cc}(c(t), C(t)) \leq 0$*,* $\forall (c(t), C(t)) \in [0, \infty) \times [0, \infty)$*, and* $\lim_{c(t) \to 0} u_c(c(t), C(t)) = \infty$*.*

Assumption 4 (the subjective discount rate function)**.** *The subjective discount rate function* $\delta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_{++}$ *is real valued, bounded above,* $\delta(0,0) = \delta_0 > n$*, and twice continuously differentiable* $\forall (c(t), C(t)) \in [0, \infty) \times [0, \infty)$*. Moreover,* $\underline{\delta} \equiv \inf \delta(c(t), C(t)) > n$*.*

Next, we define the concepts for the marginal impatience as follows. Under Assumption 4, $\forall (c(t), C(t)) \in [0, \infty) \times [0, \infty)$, it is considered that there is (i) constant marginal impatience in private consumption (resp. average consumption), if $\delta_c(c(t), C(t)) = 0$ (resp. $\delta_C(c(t), C(t)) = 0$); (ii) increasing marginal impatience in private consumption (resp. average consumption), if $\delta_c(c(t), C(t)) > 0$ (resp. $\delta_C(c(t), C(t)) > 0$); and (iii) decreasing marginal impatience in private consumption (resp. average consumption), if $\delta_c(c(t), C(t)) < 0$ (resp. $\delta_C(c(t), C(t)) < 0$).

Let us define the "social discount function" $\delta^s(c(t)) \equiv \delta(c(t), c(t))$, then

$$\delta_c^s(c(t)) = \delta_c(c(t), c(t)) + \delta_C(c(t), c(t)),$$
$$\text{and} \quad \delta_{cc}^s(c(t)) = \delta_{cc}(c(t), c(t)) + 2\delta_{cC}(c(t), c(t)) + \delta_{CC}(c(t), c(t)).$$

The "social" marginal impatience can be defined in the same manner discussed above. We make the following assumption (see also Figure 3):

Assumption 5 (the social marginal impatience)**.** *Under Assumption 4,* $\forall c(t) \in [0, \infty)$*,*

1. *when social marginal impatience is increasing, i.e.,* $\delta_c^s(c(t)) > 0$*, it holds that* $\delta_{cc}^s(c(t)) \leq 0$*, and*
2. *when social marginal impatience is decreasing, i.e.,* $\delta_c^s(c(t)) < 0$*, it holds that* $\delta_{cc}^s(c(t)) \geq 0$*.*

[Figure 3 about here.]

Furthermore, let us construct "the generating function" v,[9]

$$v\left(c\left(T\right),C\left(T\right),U\left({}_{T}E\right),n\right)\equiv u\left(c\left(T\right),C\left(T\right)\right)-U\left({}_{T}E\right)\left[\delta\left(c\left(T\right),C\left(T\right)\right)-n\right], \tag{3}$$

where

$$U\left({}_{T}E\right)\equiv\int_{T}^{\infty}u\left(c\left(t\right),C\left(t\right)\right)\exp\left[-\int_{T}^{t}\{\delta\left(c\left(\tau\right),C\left(\tau\right)\right)-n\}d\tau\right]dt.$$

We can think that the function v measures the excess of current felicity u over the "annuitised value of future utility", $U\left({}_{T}E\right)/[\delta\left(c\left(T\right),C\left(T\right)\right)-n]$.[10] At a steady state, this excess is zero. The partial derivatives of Eq. (2) are defined by using the generating function:

$$\begin{aligned}
U_{c}\left(E\right)&\equiv\frac{\partial U\left(E\right)}{\partial c\left(T\right)}=v_{c}\left(c\left(T\right),C\left(T\right),U\left({}_{T}E\right),n\right)\exp\left[-\int_{0}^{T}\{\delta\left(c\left(\tau\right),C\left(\tau\right)\right)-n\}d\tau\right],\\
U_{cc}\left(E\right)&\equiv\frac{\partial^{2}U\left(E\right)}{\partial c\left(T\right)^{2}}=v_{cc}\left(c\left(T\right),C\left(T\right),U\left({}_{T}E\right),n\right)\exp\left[-\int_{0}^{T}\{\delta\left(c\left(\tau\right),C\left(\tau\right)\right)-n\}d\tau\right],\\
U_{C}\left(E\right)&\equiv\frac{\partial U\left(E\right)}{\partial C\left(T\right)}=v_{C}\left(c\left(T\right),C\left(T\right),U\left({}_{T}E\right),n\right)\exp\left[-\int_{0}^{T}\{\delta\left(c\left(\tau\right),C\left(\tau\right)\right)-n\}d\tau\right],\\
U_{CC}\left(E\right)&\equiv\frac{\partial^{2}U\left(E\right)}{\partial C\left(T\right)^{2}}=v_{CC}\left(c\left(T\right),C\left(T\right),U\left({}_{T}E\right),n\right)\exp\left[-\int_{0}^{T}\{\delta\left(c\left(\tau\right),C\left(\tau\right)\right)-n\}d\tau\right],\\
U_{cC}\left(E\right)&\equiv\frac{\partial^{2}U\left(E\right)}{\partial C\left(T\right)\partial c\left(T\right)}=v_{cC}\left(c\left(T\right),C\left(T\right),U\left({}_{T}E\right),n\right)\exp\left[-\int_{0}^{T}\{\delta\left(c\left(\tau\right),C\left(\tau\right)\right)-n\}d\tau\right].
\end{aligned}$$

These equations imply that the first-order partial derivatives of the generating function are equal to the current-value marginal utilities of $c\left(t\right)$ and $C\left(t\right)$ of Eq. (2).

As discussed in Galí (1994) and Dupor and Liu (2003), there are two concepts about the manner in which the consumption externalities influence individual's felicity. First, for a given amount of individual consumption, the average consumption directly affects the utility level of the individual. Second, average consumption may exert an external effect on the individual's marginal utility of her/his own consumption. We apply the concepts of the consumption externalities used in the previous studies to lifetime utility. Specifically, under Assumptions 3 and 4, $\forall\left(c\left(t\right),C\left(t\right)\right)\in[0,\infty)\times[0,\infty)$, it is said to be "jealousy" if $v_{C}<0$, and "admiring" if $v_{C}>0$.[11]

[9] See Epstein (1987) for the details of the generating function.

[10] We thank Ngo Van Long for suggesting this economic implication.

[11] Furthermore, it is said to be "keeping up with the Joneses" if $v_{cC}>0$ and "running away from the Joneses"

Following Epstein and Hynes (1983), we then define the rate of time preference, $\rho(t)$, as the rate of decrease in the marginal utility of private consumption along a *locally constant* path:

$$\begin{aligned}\rho(t) &= \rho(c(t), C(t), U({}_tE), n) \\ &\equiv -\frac{d}{dt}\log\left.\frac{\partial U(E)}{\partial c(t)}\right|_{\dot{c}(t)=0} \\ &= [\delta(c(t), C(t)) - n] - \frac{v(c(t), C(t), U({}_tE), n)}{v_c(c(t), C(t), U({}_tE), n)}\delta_c(c(t), C(t)).\end{aligned} \tag{4}$$

This shows that $\rho(t)$ depends on current private consumption $c(t)$, current average consumption $C(t)$, and ${}_TE \equiv \{c(t), C(t)\}_{t=T}^{\infty}$. Thus, we can consider a rate of time preference function, $\rho(c(t), C(t), U({}_tE), n)$, where there is no harm in setting $T = 0$. We can easily check this for the time-separable preference $\rho(c(t), C(t), U({}_tE), n) = \delta_0 - n$.

Assumption 6. *Under Assumptions 3 and 4,* $\forall (c(t), C(t)) \in [0, \infty) \times [0, \infty)$*, we assume that the functions* $u(c(t), C(t))$*,* $\delta(c(t), C(t))$*, and* $v(c(t), C(t), U({}_tE), n)$ *satisfy the following conditions:*

1. $v_c(c(t), C(t), U({}_tE), n) > 0,$
   $v_{cc}(c(t), C(t), U({}_tE), n) < 0,$
   and $\lim_{c(t)\to 0} v_c(c(t), C(t), U({}_tE), n) = \infty$*;*
2. $v_c(c(t), C(t), U({}_tE), n) + v_C(c(t), C(t), U({}_tE), n) > 0,$
   $v_{cc}(c(t), C(t), U({}_tE), n) + 2v_{cC}(c(t), C(t), U({}_tE), n) + v_{CC}(c(t), C(t), U({}_tE), n) < 0,$
   and $\lim_{c(t)\to 0} [v_c(c(t), C(t), U({}_tE), n) + v_C(c(t), C(t), U({}_tE), n)] = \infty$*; and*
3. $u_c(c(t), C(t))[\delta(c(t), C(t)) - n] - u(c(t), C(t))\delta_c(c(t), C(t)) > 0.$

Assumptions 6.1 and 6.2 ensure positive marginal utility and an optimal interior solution for representative households and a social planner, respectively. Furthermore, Assumption 6.3 ensures that the time preference rate is positive. Note that Assumption 6 allows for the possibilities of either jealousy or admiration.[12]

if $v_{cC} < 0$.

[12] These assumptions are similar to those of Chen and Hsu (2007).

Assumption 7. $\frac{u_{cc}(c(t),C(t))}{u_c(c(t),C(t))} < \frac{\delta_{cc}(c(t),C(t))}{\delta_c(c(t),C(t))}$, $\forall (c(t),C(t)) \in [0,\infty) \times [0,\infty)$.

This assumption ensures that the Hamiltonian, which is specified in Eq. (6), is concave $\forall (c(t),k(t)) \in [0,\infty) \times [0,\infty)$.[13]

The flow budget constraint for the household at time t is

$$\dot{k}(t) = g(k(t),p(t)) - (\mu+n)k(t) - p(t)c(t), \tag{5}$$

where $\mu \in [0,\infty)$ denotes the rate of capital depreciation.

The consumer solves the household's maximizing problem mentioned below:

$$\begin{cases} \max_{\{c(t)\}} & \int_0^\infty u(c(t),C(t)) \exp \Delta(t) dt, \\ \text{s.t.} & \dot{k}(t) = g(k(t),p(t)) - (\mu+n)k(t) - p(t)c(t), \\ & \dot{\Delta}(t) = -[\delta(c(t),C(t)) - n], \\ & k(0) = k_0,\ \Delta(0) = 0 :\ \text{given.} \end{cases} \tag{H}$$

The present-value Hamiltonian for problem (H) is

$$\begin{aligned} &\mathcal{H}(c(t),C(t),k(t),\Delta(t),\Lambda(t),\Phi(t);p(t)) \\ &\quad \equiv u(c(t),C(t))\exp\Delta(t) + \Lambda(t)[g(k(t),p(t)) - (\mu+n)k(t) - p(t)c(t)] - \Phi(t)[\delta(c(t),C(t)) - n], \end{aligned} \tag{6}$$

where $\Lambda(t)$ and $\Phi(t)$ are the costate variables of $k(t)$ and $\Delta(t)$, respectively. The first-order conditions of this problem are given by

$$p(t)\lambda(t) = v_c(c(t),C(t),\phi(t),n), \tag{7a}$$

$$\dot{\lambda}(t) = [\delta(c(t),C(t)) - \{g_k(k(t),p(t)) - \mu\}]\lambda(t), \tag{7b}$$

$$\dot{\phi}(t) = -v(c(t),C(t),\phi(t),n), \tag{7c}$$

$$\dot{k}(t) = g(k(t),p(t)) - (\mu+n)k(t) - p(t)c(t), \tag{7d}$$

$$\lim_{t\to\infty} k(t)\lambda(t)\exp\Delta(t) = 0, \tag{7e}$$

[13]This assumption is similar to Palivos et al. (1997; Assumption ($3iii$)).

where $\lambda(t)$ and $\phi(t)$ are the current-values of $\Lambda(t)$ and $\Phi(t)$, respectively. In other words, $\lambda(t) \equiv \Lambda(t) \exp[-\Delta(t)]$ and $\phi(t) \equiv \Phi(t) \exp[-\Delta(t)]$. Note that $\phi(T) = U({}_T E)$, that is, the costate variable $\phi(T)$ represents the lifetime utility after time T.[14]

4. Dynamic System

This section specifies a complete dynamic system of a small open economy in order to discuss the *existence*, *uniqueness*, and *stability* of long-run equilibria. We assume that the small open economy faces the constant world price over time, and that there exists a continuum of identical households distributed uniformly over $[0, 1]$.

Definition 1. *Under Assumptions 1–7, for k_0, a symmetric market equilibrium in a small open economy is a path $\{c(t), \lambda(t), \phi(t), k(t)\}$ with $C(t) = c(t)$, $\forall t \in [0, \infty)$, which solves Eq.(7).*

Assumption 8. *$p(t) = p$, $\forall t \in [0, \infty)$.*

The complete dynamic system with respect to c, ϕ, and k is constituted by the following differential equations:

$$\dot{c}(t) = \sigma(c(t), \phi(t)) [\{g_k(k(t), p) - (\mu + n)\} - \rho(c(t), c(t), \phi(t), n)] c(t), \tag{8a}$$

$$\dot{k}(t) = g(k(t), p) - (\mu + n) k(t) - pc(t), \tag{8b}$$

$$\dot{\phi}(t) = -[u(c(t), c(t)) - \phi(t) \{\delta(c(t), c(t)) - n\}], \tag{8c}$$

where

$$\sigma(c(t), \phi(t)) \equiv -\frac{v_c(c(t), c(t), \phi(t), n)}{[v_{cc}(c(t), c(t), \phi(t), n) + v_{cC}(c(t), c(t), \phi(t), n)] c(t)}.$$

We make an following assumption so as to obtain positive $\sigma(c(t), \phi(t))$.

Assumption 9. *$v_{cc}(c(t), c(t), \phi(t), n) + v_{cC}(c(t), c(t), \phi(t), n) < 0$, $\forall (c(t), \phi(t)) \in [0, \infty) \times (-\infty, \infty)$.*

[14] For details of the derivation, see Karasawa (2007).

This assumption says that the marginal social utility of consumption is positive, i.e.,

$$\frac{d}{dc(t)}\left[v_c\left(c\left(t\right), c\left(t\right), \phi\left(t\right), n\right) + v_C\left(c\left(t\right), c\left(t\right), \phi\left(t\right), n\right)\right] > 0.$$

A detailed economic interpretation of this assumption is given in Fisher and Hof (2000, p.247), who stated that "Loosely speaking, this condition ensures that individuals do not *overreact* to changes in average consumption and that equilibrium consumption still depends negatively on λ, as in the standard model."

5. Long-Run Equilibrium

The steady state of the dynamic system is realized when all variables in (8) stay constant over time:

$$\dot{c}\left(t\right) = \dot{k}\left(t\right) = \dot{\phi}\left(t\right) = 0.$$

Let us denote the steady state values of the variables with "$-$." In the steady state, we can derive $\rho\left(\bar{c}, \bar{c}, \bar{\phi}, n\right) = \delta\left(\bar{c}, \bar{c}\right) - n$ because $\dot{\phi}\left(t\right) = -v\left(\bar{c}, \bar{c}, \bar{\phi}, n\right) = -\left\{u\left(\bar{c}, \bar{c}\right) - \bar{\phi}\left[\delta\left(\bar{c}, \bar{c}\right) - n\right]\right\} = 0$. The steady state $\left(\bar{c}, \bar{k}, \bar{\phi}\right)$ is characterized by the following set of conditions:

$$g_k\left(\bar{k}, p\right) - \left(\mu + n\right) = \delta\left(\bar{c}, \bar{c}\right) - n, \tag{9a}$$

$$g\left(\bar{k}, p\right) = \left(\mu + n\right)\bar{k} + p\bar{c}, \tag{9b}$$

$$\bar{\phi} = \frac{u\left(\bar{c}, \bar{c}\right)}{\delta\left(\bar{c}, \bar{c}\right) - n}. \tag{9c}$$

Once we find a solution for $\bar{c}$, we can uniquely derive the lifetime utility $\bar{\phi}$ from Eq. (9c). Furthermore, using (9b), the steady-state value of per capita consumption $\bar{c}$ is given by

$$\bar{c} = \frac{g\left(\bar{k}, p\right) - \left(\mu + n\right)\bar{k}}{p}.$$

Therefore, we can then derive the following reduced-form equation characterizing the steady state in terms of a single variable $\bar{k}$:

$$g_k\left(\bar{k}, p\right) = \delta\left(\frac{g\left(\bar{k}, p\right) - \left(\mu + n\right)\bar{k}}{p}, \frac{g\left(\bar{k}, p\right) - \left(\mu + n\right)\bar{k}}{p}\right) + \mu. \tag{10}$$

The steady-state condition (10) can be interpreted as a market clearing condition for capital in the long run. The left-hand side of Eq. (10) denotes the value of the marginal product of capital, which, as noted in Section 2, defines the demand function for capital determined by competitive firms:

$$r = g_k(k, p), \tag{11}$$

with the slope

$$g_{kk} \begin{cases} < 0, \text{ if } k \in [0, k_1(p)] \cup [k_2(p), \infty), \\ = 0, \text{ if } k \in (k_1(p), k_2(p)). \end{cases}$$

The right-hand side of Eq. (10) denotes the sum of discount and depreciation rates, which can be interpreted as the households' supply function of capital in the long-run:

$$r = \delta\left(\frac{g(k,p) - (\mu+n)k}{p}, \frac{g(k,p) - (\mu+n)k}{p}\right) + \mu \equiv \tilde{\delta}(k, p, n) + \mu, \tag{12}$$

with the slope

$$(\delta_c + \delta_C)\left[\frac{g_k - (\mu+n)}{p}\right] = (\delta_c + \delta_C)\left[\frac{\delta - n}{p}\right] \begin{cases} > 0, \text{ if } \delta_c + \delta_C > 0, \\ = 0, \text{ if } \delta_c + \delta_C = 0, \\ < 0, \text{ if } \delta_c + \delta_C < 0. \end{cases}$$

In analyzing the *existence* and *uniqueness* of the steady state, a comparison of the slopes of the demand and supply functions derived above plays a key role.

In his one-sector optimal growth model in a closed economy, Chang (2009) assumed the "bounded slope assumption" to ensure the existence of the steady state even if the discount function exhibits decreasing marginal impatience.[15] We impose the following assumption as a corresponding condition to Chang's.

Assumption 10 (bounded slope of the function $\delta(c(t), C(t))$).

$$\delta_c(0,0) + \delta_C(0,0) \geq \left(\frac{p}{\delta_0}\right) \max\left\{pf_1''(k), f_2''(k)\right\}, \quad \forall k \in \left[\underline{\underline{k}}, \overline{\overline{k}}\right],$$

where $\underline{\underline{k}} \equiv \min\left\{pf_1'^{-1}((\delta_0+\mu)/p), f_2'^{-1}(\delta_0+\mu)\right\}$ *and* $\overline{\overline{k}} \equiv \max\left\{pf_1'^{-1}((\underline{\delta}+\mu)/p), f_2'^{-1}(\underline{\delta}+\mu)\right\}$,

[15] For the economic implication of this condition, see Chang (2009).

$\forall p \in (0, \infty)$.

We can derive the following proposition with regard to the long-run equilibrium.

Proposition 1.

1. *If* $\delta_c + \delta_C = 0$, *then*

   a. *there are a continuum of possible equilibria with diversified production and two possible equilibria specified to either good (i.e., equilibria that result in the capital stock* $\bar{k} \in [k_1(p), k_2(p)]$*) at only* $p = \hat{p}$, *which satisfies* $\hat{r}(\hat{p}) = \delta_0 + \mu$, *where* $\hat{r}(p)$ *is the rental rate of capital under incomplete specialization,*

   b. *there is a unique equilibrium specialized to either good* $\forall p \in (0, \hat{p}) \cup (\hat{p}, \infty)$;

2. *if* $\delta_c + \delta_C > 0$, *then there is a unique equilibrium* $\forall p \in (0, \infty)$;

3. *if* $\delta_c + \delta_C < 0$ *and we denote an open set* $\Omega \subset (0, \infty)$ *as the set of* p *that can satisfy incomplete specialization, then*

   a. *there are three possible equilibria, that is,* $\bar{k} \in (k_1(p), k_2(p))$, $\bar{k} \in (0, k_1(p))$, *and* $\bar{k} \in (k_2(p), \infty)$, $\forall p \in \Omega$,

   b. *there are two possible equilibria, that is,* $\bar{k} = k_1(p)$ *and* $\bar{k} \in (k_2(p), \infty)$ *(resp.* $\bar{k} = k_2(p)$ *and* $\bar{k} \in (0, k_1(p))$*) at* $p = P_1$ *(resp.* $p = P_2$*), where* $P_i \equiv \{p : \bar{k} = k_i(p)\}$, $i = 1, 2$,

   c. *there is a unique long-run equilibrium specialized to either good* $\forall p \notin \Omega \cup \{P_1\} \cup \{P_2\}$.

Proof. See Appendix B. □

A diagrammic representation of the situation 3a in Proposition 1 is given in Figure 4. In this figure, there is an incomplete specialization equilibrium and two other equilibria that are specialized to either good at $p \in \Omega$.

[Figure 4 about here.]

6. Transitional Dynamics and Local Indeterminacy

Let us analyze the local stability property of the steady states. Linearizing the system of differential equations (8) around the steady state yields

$$\begin{bmatrix} \dot{c} \\ \dot{\phi} \\ \dot{k} \end{bmatrix} = \begin{bmatrix} \frac{v_c \rho_C}{v_{cc}+v_{cC}} & \frac{(\delta-n)\delta_c}{v_{cc}+v_{cC}} & -\frac{v_c g_{kk}}{v_{cc}+v_{cC}} \\ -(v_c+v_C) & \delta-n & 0 \\ -p & 0 & \delta-n \end{bmatrix} \begin{bmatrix} c-\bar{c} \\ \phi-\bar{\phi} \\ k-\bar{k} \end{bmatrix}, \tag{13}$$

where $\rho_C \equiv (v_c\delta_C - v_C\delta_c)/v_c = \delta_C - v_C\delta_c/v_c \gtreqless 0$. Evaluating the Jacobian of the above system at the steady state, we obtain the following characteristic equation:

$$J(\xi_i) \equiv -\xi_i^3 + A\xi_i^2 + B\xi_i + C = 0,$$

where ξ_i $(i = 1, 2, 3)$ denotes the eigenvalues and

$$\begin{aligned} A &\equiv \left[2(\delta-n) + \frac{v_c\rho_C}{v_{cc}+v_{cC}}\right], \\ B &\equiv -(\delta-n)\left[(\delta-n) + \frac{v_c\rho_C}{v_{cc}+v_{cC}}\right] + \frac{v_c\{pg_{kk} - (\delta-n)(\delta_c+\delta_C)\}}{v_{cc}+v_{cC}}, \\ C &\equiv \frac{v_c(\delta-n)\{pg_{kk} - (\delta-n)(\delta_c+\delta_C)\}}{v_{cc}+v_{cC}}. \end{aligned}$$

By factorizing this characteristic equation, we get

$$J(\xi_i) = -[\xi_i - (\delta-n)]\left[\xi_i^2 - \left\{(\delta-n) + \frac{v_c\rho_C}{v_{cc}+v_{cC}}\right\}\xi_i + \frac{v_c\{(\delta-n)(\delta_c+\delta_C) - pg_{kk}\}}{v_{cc}+v_{cC}}\right] = 0.$$

One of the characteristic roots, say ξ_1, is given by $\delta - n > 0$. The other characteristic roots, say ξ_2 and ξ_3 $(\xi_2 < \xi_3)$, are the solutions of the following quadratic equation $\Psi(\xi_i)$:

$$\Psi(\xi_i) \equiv \xi_i^2 - \left\{(\delta-n) + \frac{v_c\rho_C}{v_{cc}+v_{cC}}\right\}\xi_i + \frac{v_c\{(\delta-n)(\delta_c+\delta_C) - pg_{kk}\}}{v_{cc}+v_{cC}} = 0.$$

Therefore, it follows that

$$\xi_2 + \xi_3 = (\delta - n) + \frac{v_c \rho_C}{v_{cc} + v_{cC}} \gtreqless 0,$$
$$\xi_2 \xi_3 = \bar{c}\sigma \{ p g_{kk} - (\delta - n)(\delta_c + \delta_C) \} \gtreqless 0.$$

If $\xi_2\xi_3 < 0$, the characteristic equation has one negative and two positive roots. Since the dynamical system has one predetermine variable k, this imples that the steady state is a local saddle point. If $\xi_2\xi_3 > 0$, two cases are possible: one where all characteristic roots have positive real parts ($\xi_2 + \xi_3 > 0$) and the other where both ξ_2 and ξ_3 have negative real parts ($\xi_2 + \xi_3 < 0$). In the former case, the steady state is locally unstable, while in the latter, the dynamic equilibrium around the steady state is locally indeterminate. Given (11) and (12), the sign of $pg_{kk} - (\delta - n)(\delta_c + \delta_C)$ depends on the slope of the curves that characterize the demand and supply functions for capital. That is, if the slope of the supply function is larger (resp. smaller) than that of the demand function, $\xi_2\xi_3 < 0$ (resp. $\xi_2\xi_3 > 0$).

Proposition 2. *Around the steady state,*

1. *if* $(\delta - n)(\delta_c + \delta_C) > pg_{kk} \Leftrightarrow (\delta_c + \delta_C)\left(\frac{\delta - n}{p}\right) > g_{kk}$, *that is, the slope of the supply function is larger than that of the demand function, the steady state is locally saddle-point stable;*

2. *if* $(\delta - n)(\delta_c + \delta_C) < pg_{kk} \leq 0 \Leftrightarrow (\delta_c + \delta_C)\left(\frac{\delta - n}{p}\right) < g_{kk} \leq 0$, *that is, the slope of the supply function is smaller than that of the demand function and*

   a. $\rho_C < -\frac{(\delta - n)(v_{cc} + v_{cC})}{v_c}$, *the steady state is locally unstable,*

   b. $\rho_C > -\frac{(\delta - n)(v_{cc} + v_{cC})}{v_c} > 0$, *the steady state is locally indeterminate.*

Proof. See Appendix C. □

From the above analysis, if $\delta_c + \delta_C > 0$, which satisfies the situation 1 of Proposition 2, the unique long-run equilibrium (see Proposition 1) is locally saddle-point stable, regardless of the value of relative price p.[16]

[16] The stability analysis for $\delta_c + \delta_C = 0$ is fundamentally similar to that in Karasawa (2007).

If $\delta_c + \delta_C < 0$, the situation 2 of Proposition 2 may occur. Suppose that there are three steady states and consider the situation 2a of Proposition 2. Then, from the initial capital stock $k_0 \neq \bar{k}_d(p)$, $\forall p \in \Omega$, the economy converges to the steady state specialized to either good, say E_1 or E_2 in Figure 4, over time. In this case, equilibria E_1 and E_2 are locally saddle-point stable since the slope of the supply function is larger than that of the demand function, whereas the diversified equilibrium E_d, in which the slope of the supply function is smaller than that of the demand function, is locally unstable. As is well known, this phenomenon is known as the "poverty trap."[17] Poor countries with low initial endowments of capital converge to a low steady state, while rich countries converge to a high one, even though all countries share identical technologies and preferences in this setting.

On the other hand, if $\delta_c + \delta_C < 0$ and the situation 2b of Proposition 2, are satisfied $\forall p \in \Omega$, the initial capital stock k_0 and Eq. (13) cannot determine whether the economy will be completely or incompletely specialized in the long-run. While equilibria E_1 and E_2 are locally saddle-point stable, diversified equilibrium E_d is locally indeterminate in this case. For the existence of locally indeterminate equilibria, we need a large degree of increasing time preference in average consumption and a sufficiently large degree of decreasing impatience in private consumption so as to dominate the increasing impatience in the average consumption. The implication of local indeterminacy is that there will be multiple paths converging to a given steady state, which is equilibrium E_d in this case. Hence, indeterminacy guarantees the existence of a continuum of sunspot stationary equilibria, i.e., stochastic rational expectations equilibria determined by perturbations unrelated to the uncertainty in economic fundamentals.

Why, then, can indeterminacy occur in our model if $\rho_C > -(\delta - n)(v_{cc} + v_{cC})/v_c > 0$ and at the same time $\delta_c + \delta_C < 0$? To answer this, let us consider the economy initially at an equilibrium path. Suppose that the agent has expectations that the average consumption in this economy is higher. As the agents' time preference is an increasing function with respect to average consumption, that is, $\rho_C > 0$ and the degree of increase is very large, if the average consumption level is higher, each agent increases her/his consumption level significantly at the present because the discount rate applied to the future utility will be higher. On the other hand, δ can interpreted as the marginal cost of capital accumulation, as seen in Eq. (12). Moreover,

[17] Deardorff (2001) analyzes a multisector neoclassical growth model in a small open economy and, by assuming savings out of wages, demonstrates multiple steady states and poverty trap.

with incomplete specialization that implies a constant rate of capital return for the small open economy, if $\delta_c + \delta_C < 0$, the agent will accumulate her/his capital further because the net return of capital $\delta - n$ increases. As the agents' income will increase with capital accumulation, the expectation with regard to higher consumption by other agents will lead to an increase in private consumption. Therefore, the new equilibrium path is self-fulfilling.

In order to further interpret the conditions for indeterminacy, let us consider two special cases: (i) the consumption externality exists only in the discount rate function (i.e., $u_C = 0$) and (ii) the consumption externality exists only in the felicity function (i.e., $\delta_C = 0$). In case (i), ρ_C is reduced to δ_C. Therefore, for the dynamic equilibrium of the economy to exhibit indeterminacy, we need a large degree of increasing impatience in average consumption, and given the other condition $\delta_c + \delta_C < 0$, we also need a sufficiently large degree of decreasing impatience in private consumption so as to dominate the effect of δ_C. In case (ii), $\rho_C = -v_C \delta_c / v_c$ and the condition $\delta_c + \delta_C < 0$ is reduced to δ_c. Therefore, in this case, indeterminacy can arise if individuals feel admiration and if there is decreasing marginal impatience in private consumption. These conditions are essentially the same as those in the closed-economy model examined by Chen and Hsu (2007).

7. The Long-Run Effects of a Terms-of-Trade Deterioration

In this section, we analyze the steady-state effect of an increase in the relative price p in the case of $p \in \Omega$. Differentiating Eqs. (9a) and (9b) at $dn = 0$, we obtain the following system:

$$\begin{bmatrix} \delta_c + \delta_C & -g_{kk} \\ p & -[g_k - (\mu + n)] \end{bmatrix} \begin{bmatrix} d\bar{c} \\ d\bar{k} \end{bmatrix} = \begin{bmatrix} g_{kp} \\ g_p - \bar{c} \end{bmatrix} dp. \tag{14}$$

Therefore, the steady state effects of an increase in p, $d\bar{c}/dp$ and $d\bar{k}/dp$, are obtained as follows:

$$\begin{cases} \dfrac{d\bar{c}}{dp} = \dfrac{[g_k - (\mu + n)]\, g_{kp}}{(\delta_c + \delta_C)[g_k - (\mu + n)] - p g_{kk}} + (g_p - \bar{c}) \dfrac{-g_{kk}}{(\delta_c + \delta_C)[g_k - (\mu + n)] - p g_{kk}}, \\ \dfrac{d\bar{k}}{dp} = \dfrac{p g_{kp}}{(\delta_c + \delta_C)[g_k - (\mu + n)] - p g_{kk}} + (g_p - \bar{c}) \dfrac{-(\delta_c + \delta_C)}{(\delta_c + \delta_C)[g_k - (\mu + n)] - p g_{kk}}. \end{cases}$$

Furthermore, $\forall p \in \Omega$, we can derive the following effects because $g_k = \hat{r}(p)$, $g_{kk} = 0$, and $g_{kp} = \hat{r}'(p)$:

$$\frac{d\bar{c}}{dp} = \frac{\hat{r}'(p)}{(\delta_c + \delta_C)}, \tag{15a}$$

$$\frac{d\bar{k}}{dp} = \frac{p\hat{r}'(p)}{(\delta_c + \delta_C)\left[\hat{r}(p) - (\mu + n)\right]} + (g_p - \bar{c})\frac{-1}{\left[\hat{r}(p) - (\mu + n)\right]}. \tag{15b}$$

Eq. (15b) is a simple exercise of basic microeconomic theory. The first term on the right-hand side measures the substitution effect, which is negative when $(\delta_c + \delta_C) > 0$ and $k_1(p) < k_2(p)$. Moreover, the second term measures the income effect and is negative (resp. positive) when $g_p > \bar{c}$ (resp. $g_p < \bar{c}$). In the following analysis, we focus on the case where the income effect does not dominate the substitution effect.

Moreover, differentiating Eq. (9c) at $dn = 0$, the steady-state effect of an increase in p on long-run lifetime welfare $\bar{\phi}$ is obtained as follows:

$$d\bar{\phi} = \frac{v_c\left(\bar{c}, \bar{c}, \bar{\phi}\right)}{\left[\delta(\bar{c}, \bar{c}) - n\right]} d\bar{c} = \frac{v_c\left(\bar{c}, \bar{c}, \bar{\phi}\right)}{\left[\hat{r}(p) - (\mu + n)\right]} d\bar{c}.$$

Therefore, we can derive the following long-run effect of an increase in p:

$$\frac{d\bar{\phi}}{dp} = \frac{v_c\left(\bar{c}, \bar{c}, \bar{\phi}\right)}{\left[\hat{r}(p) - (\mu + n)\right]} \cdot \frac{d\bar{c}}{dp} = \frac{\hat{r}'(p)\, v_c\left(\bar{c}, \bar{c}, \bar{\phi}\right)}{(\delta_c + \delta_C)\left[\hat{r}(p) - (\mu + n)\right]}. \tag{15c}$$

We thus have the following result:

Proposition 3. *Consider the equilibrium $\bar{k} \in (k_1(p), k_2(p))$, $\forall p \in \Omega$.*

1. *If $\delta_c + \delta_C > 0$ evaluated at a steady state, then*

$$\frac{d\bar{c}}{dp} < 0, \qquad \frac{d\bar{k}}{dp} < 0, \qquad \frac{d\bar{\phi}}{dp} < 0.$$

2. *By contrast, if $\delta_c + \delta_C < 0$ evaluated at a steady state, then*

$$\frac{d\bar{c}}{dp} > 0, \qquad \frac{d\bar{k}}{dp} > 0, \qquad \frac{d\bar{\phi}}{dp} > 0.$$

Given Proposition 2, Proposition 3 suggests that the comparative static effects of a change in the relative price can be different for saddle-point stability and indeterminacy. As analyzed by Nishimura and Shimomura (2002a), this may also affect the properties of the economy's specialization pattern in the long run, even though we do not consider international equilibrium with two large countries but a small open economy facing a given world price.

8. Concluding Remarks

In this paper, we present a dynamic small country model of international trade with variable marginal impatience and consumption externalities.

For socially increasing marginal impatience, the steady state with incomplete specialization exhibits uniqueness and saddle-point stability. On the other hand, for socially decreasing marginal impatience, the steady state with incomplete specialization does not exhibit uniqueness and saddle-point stability. Specifically, we derive indeterminacy of the dynamic equilibrium for socially decreasing marginal impatience.

We conclude the paper by suggesting directions for further research. First, we need to analyze the relationship between the trade equilibrium, in which the world commodity prices are determined endogenously, and the variable, especially decreasing, marginal impatience. Second, it would be interesting to discuss about the gains from trade in the current framework. Third, we need to derive the effect of the trade policy. Finally, it is important to extend the current framework to the endogenous growth theory. These are the problems that are yet to be examined.

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Appendix A. Illustration of the GDP function

As long as both goods are produced, it follows from Assumption 2 that $k_i\left(p\left(t\right)\right)$ in Figure 1 has the following property:

$$\frac{dk_i\left(p\left(t\right)\right)}{dp\left(t\right)} > 0,\ i = 1, 2,\ \forall p\left(t\right) \in \left(0, \infty\right). \tag{A1}$$

The labor allocations $l_i\left(t\right)$ is then obtained from resource constraints,

$$\begin{aligned} l_1\left(t\right) \equiv l_1\left(k\left(t\right), p\left(t\right)\right) &= \frac{k\left(t\right) - k_2\left(p\left(t\right)\right)}{k_1\left(p\left(t\right)\right) - k_2\left(p\left(t\right)\right)}, \\ l_2\left(t\right) \equiv l_2\left(k\left(t\right), p\left(t\right)\right) &= \frac{k_1\left(p\left(t\right)\right) - k\left(t\right)}{k_1\left(p\left(t\right)\right) - k_2\left(p\left(t\right)\right)}. \end{aligned}$$

From Assumptions 1, 2, and the properties of GDP functions , given the capital-labor ratio $k\left(t\right)$, we obtain the pattern of the specialization as follows:

$$\begin{cases} 0 < k\left(t\right) \le k_1\left(p\left(t\right)\right), & \text{specialized to the good 1,} \\ k_1\left(p\left(t\right)\right) < k\left(t\right) < k_2\left(p\left(t\right)\right), & \text{incompletely specialized,} \\ k_2\left(p\left(t\right)\right) \le k\left(t\right) < \infty, & \text{specialized to the good 2,} \end{cases}$$

$\forall p\left(t\right) \in \left(0, \infty\right)$. Furthermore, by using $l_1\left(t\right) k_1\left(t\right) + l_2\left(t\right) k_2\left(t\right) \le k\left(t\right)$ and Eq. (A1), the pattern of the specialization is reduced to

$$\begin{cases} 0 < p\left(t\right) \le p_2\left(k\left(t\right)\right), & \text{specialized to the good 2,} \\ p_2\left(k\left(t\right)\right) < p\left(t\right) < p_1\left(k\left(t\right)\right), & \text{incompletely specialized,} \\ p_1\left(k\left(t\right)\right) \le p\left(t\right) < \infty, & \text{specialized to the good 1,} \end{cases}$$

where $p_i\left(k\left(t\right)\right) \equiv k_i^{-1}\left(k\left(t\right)\right)$, $i = 1, 2$, for $\forall k\left(t\right) \in \left[0, \infty\right)$. Finally, we can define the open set of $\left(k\left(t\right), p\left(t\right)\right)$ for diversified production as follows:

$$\Gamma \equiv \left\{\left(k\left(t\right), p\left(t\right)\right) \in \mathbb{R}_{++} \times \mathbb{R}_{++} : k\left(t\right) \in \left(k_1\left(p\left(t\right)\right), k_2\left(p\left(t\right)\right)\right)\right\}.$$

This set is illustrated by the shaded region between schedules $k_1\left(p\left(t\right)\right)$ and $k_2\left(p\left(t\right)\right)$ in $\left(k\left(t\right), p\left(t\right)\right)$ plane of Figure 1. This economy produces both goods only if the relative price $p\left(t\right)$ and the capital-labor ratio $k(t)$ belong to the region.

Let $\hat{r}(t)$ be the rental rate of capital under incomplete specialization, then it is the slope of tangent AB in Figure 1 and has the properties

$$\begin{aligned} \hat{r}(t) \equiv \hat{r}(p(t)) = p(t) f_1'(k_1(p(t))) = f_2'(k_2(p(t))), \\ \text{where } \hat{r}'(p(t)) < 0, \ \forall p(t) \in (p_2(k(t)), p_1(k(t))). \end{aligned} \tag{1c$'$}$$

Furthermore, let $\hat{w}(t)$ be the wage rate under incomplete specialization, then it is the vertical intercept of tangent AB in Figure 1 and has the properties

$$\begin{aligned} \hat{w}(t) \equiv \hat{w}(p(t)) &= p(t)\left[f_1(k_1(p(t))) - k_1(p(t)) f_1'(k_1(p(t)))\right] \\ &= f_2(k_2(p(t))) - k_2(p(t)) f_2'(k_2(p(t))), \\ &\text{where } \hat{w}'(p(t)) > 0, \ \forall p(t) \in (p_2(k(t)), p_1(k(t))). \end{aligned} \tag{1d$'$}$$

By using $\hat{r}(p(t))$, $\hat{w}(p(t))$, and (1c$'$), we have

$$y_1(k(t), p(t)) = \hat{r}'(p(t)) k(t) + \hat{w}'(p(t)), \tag{1a$'$}$$

$$y_2(k(t), p(t)) = \left[\hat{r}(p(t)) - p(t)\hat{r}'(p(t))\right] k(t) + \left[\hat{w}(p(t)) - p(t)\hat{w}'(p(t))\right], \tag{1b$'$}$$

$$p(t) y_1(k(t), p(t)) + y_2(k(t), p(t)) = \hat{r}(p(t)) k(t) + \hat{w}(p(t)), \tag{1e$'$}$$

under incomplete specialization.

Appendix B. Proof of Proposition 1

Before proving Proposition 1, we show the properties of the steady state. Note that as shown of Figure 5, we can confine the analysis to the region in which $k \in [0, k_c(p,n)]$, where $k_c(p,n)$ is the capital stock level at which output per head is just sufficient to replace depreciation per head, that is, $k_c(p,n) \equiv \{k : g(k,p) = (\mu + n) k, \ k > 0\}$, because of the interior solution about c.

[Figure 5 about here.]

The function $g_k(k,p)$ has the following properties:

$$\lim_{k\to 0} g_k(k,p) = \infty,$$
$$g_k(k_g(p,n),p) = n+\mu,$$
$$g_k(k,p) = \hat{r}(p), \quad k \in [k_1(p), k_2(p)],$$
$$g_{kk}(k,p)\begin{cases} <0, & \text{if } k \in [0,k_1(p)] \cup [k_2(p),\infty), \\ =0, & \text{if } k \in (k_1(p), k_2(p)), \end{cases}$$

where $k_g(p,n) \in (0, k_c(p,n))$ is golden rule capital stock. Moreover, the properties of $g_{kp}(k,p)$ are characterized by the following lemmas.

Lemma 1 (Rybczynski)**.** *Suppose that* $\mathbf{y}(t) \gg 0$*, Assumption 1, and 2 are satisfied. If* $k_2(p(t)) > k_1(p(t))$*,* $\forall p(t) \in (p_2(k(t)), p_1(k(t)))$*, then*

$$\frac{\partial y_1(t)}{\partial k(t)} = g_{pk}(k(t),p(t)) < 0.$$
$$\frac{\partial y_2(t)/\partial k(t)}{y_2(t)/k(t)} = \frac{k(t)[g_k(k(t),p(t)) - p(t)g_{pk}(k(t),p(t))]}{g(k(t),p(t)) - p(t)g_p(k(t),p(t))} > 1.$$

Lemma 2 (Stolper–Samuelson)**.** *Suppose that* $\mathbf{y}(t) \gg 0$*, Assumption 1, and 2 are satisfied. If* $k_2(p(t)) > k_1(p(t))$*,* $\forall p(t) \in (p_2(k(t)), p_1(k(t)))$*, then*

$$\frac{dr(t)}{dp(t)} = g_{kp}(k(t),p(t)) < 0,$$
$$\frac{dw(t)/dp(t)}{w(t)/p(t)} = \frac{p(t)[g_p(k(t),p(t)) - k(t)g_{kp}(k(t),p(t))]}{g(k(t),p(t)) - k(t)g_k(k(t),p(t))} > 1.$$

Lemmas 1 and 2 provide the properties of $g_{kp}(k(t),p(t))$ under incomplete specialization.[18] By contrast, we obtain the following lemma as the properties of $g_{kp}(k(t),p(t))$ under complete specialization in good 1 or good 2, because $g(k(t),p(t)) = f_2(k(t)), \forall p(t) \in (0, p_2(k(t))]$ and $g(k(t),p(t)) = p(t)f_1(k(t)), \forall p(t) \in [p_1(k(t)), \infty)$.

Lemma 3. *Suppose that Assumptions 1 and 2 are satisfied.*

1. $\forall p(t) \in (0, p_2(k(t))]$, $g_{kp}(k(t),p(t)) = \dfrac{\partial^2 f_2(k(t))}{\partial k(t)\,\partial p(t)} = 0.$

2. $\forall p(t) \in [p_1(k(t)), \infty)$,
$$g_{kp}(k(t),p(t)) = \frac{\partial}{\partial p(t)}\left[\frac{\partial p(t) f_1(k(t))}{\partial k(t)}\right] = \frac{\partial^2 p(t) f_1(k(t))}{\partial k(t)\,\partial p(t)} = f_1'(k(t)) > 0.$$

[18] As is well known, these lemmas show the *magnification effects*. On comparing Lemma 1 with 2, we observe the fact that these properties are equivalent, that is the *reciprocity relation*.

The properties of function $\tilde{\delta}(k,p,n)+\mu$ are as follows:

$$\lim_{k\to 0}\tilde{\delta}(k,p,n)+\mu=\tilde{\delta}(k_c(p,n),p,n)+\mu=\delta_0+\mu,$$

$$\frac{\partial\left(\tilde{\delta}(k,p,n)+\mu\right)}{\partial k}=\frac{\delta_c+\delta_C}{p}\left[g_k-(\mu+n)\right]
\begin{cases}=0, & \text{if } \delta_c+\delta_C=0,\\ \begin{cases}>0, & \text{if } \delta_c+\delta_C>0 \text{ and } k\in[0,k_g(p,n)),\\ =0, & \text{if } \delta_c+\delta_C>0 \text{ and } k=k_g(p,n),\\ <0, & \text{if } \delta_c+\delta_C>0 \text{ and } k\in(k_g(p,n),k_c(p,n)),\end{cases}\\ \begin{cases}<0, & \text{if } \delta_c+\delta_C<0 \text{ and } k\in[0,k_g(p,n)),\\ =0, & \text{if } \delta_c+\delta_C<0 \text{ and } k=k_g(p,n),\\ >0, & \text{if } \delta_c+\delta_C<0 \text{ and } k\in(k_g(p,n),k_c(p,n)),\end{cases}\end{cases}$$

$$\frac{\partial^2\left(\tilde{\delta}(k,p,n)+\mu\right)}{\partial k^2}=\frac{(\delta_{cc}+2\delta_{cC}+\delta_{CC})\left[g_k-(\mu+n)\right]^2+(\delta_c+\delta_C)pg_{kk}}{p^2}
\begin{cases}=0, & \text{if } \delta_c+\delta_C=0,\\ <0, & \text{if } \delta_c+\delta_C>0,\\ >0, & \text{if } \delta_c+\delta_C<0,\end{cases}$$

$$\frac{\partial\left(\tilde{\delta}(k,p,n)+\mu\right)}{\partial p}=\frac{(\delta_c+\delta_C)\left[(\mu+n)k-\{g-pg_p\}\right]}{p^2}
\begin{cases}=0, & \text{if } \delta_c+\delta_C=0,\\ \begin{cases}>0, & \text{if } \delta_c+\delta_C>0 \text{ and } (\mu+n)k-\{g-pg_p\}>0,\\ =0, & \text{if } \delta_c+\delta_C>0 \text{ and } (\mu+n)k-\{g-pg_p\}=0,\\ <0, & \text{if } \delta_c+\delta_C>0 \text{ and } (\mu+n)k-\{g-pg_p\}<0,\end{cases}\\ \begin{cases}<0, & \text{if } \delta_c+\delta_C<0 \text{ and } (\mu+n)k-\{g-pg_p\}>0,\\ =0, & \text{if } \delta_c+\delta_C<0 \text{ and } (\mu+n)k-\{g-pg_p\}=0,\\ >0, & \text{if } \delta_c+\delta_C<0 \text{ and } (\mu+n)k-\{g-pg_p\}<0,\end{cases}\end{cases}$$

where $(\mu+n)k-\{g-pg_p\}$ is the net import of investment goods in the steady state.

Lemma 4. *There is no the steady state in the region in which* $k\in[k_g(p,n),k_c(p,n)]$.

Proof. By the analysis above, $\tilde{\delta}(k,p,n)\geq\underline{\delta}>n$, in which $k\in[0,k_c(p,n)]$. On the other hand, $g_k(k,p)\leq n+\mu$, in which $k\in[k_g(p,n),k_c(p,n)]$. Therefore, $\tilde{\delta}(k,p,n)+\mu>g_k(k,p)$ in which $k \in [k_g(p,n), k_c(p,n)]$ necessarily. □

Using Lemma 1 – 4, we can prove Proposition 1 as follows.

B.1. Proof of Proposition 1.1a

First, if $\delta_c+\delta_C=0$, then $\tilde{\delta}(k,p,n)+\mu=\delta_0+\mu$, $\forall k \in [0, k_g(\hat{p},n))$, $\forall p \in (0,\infty)$, and $\forall n \in [0,\infty)$. Second, $g_k(k,\hat{p})$ is a continuous and monotonic decreasing function with respect to k, and has the following properties:

$$g_k(k,\hat{p}) \begin{cases} > \delta_0+\mu, & \forall k \in [0, k_1(\hat{p})), \\ = g_k(k_1(\hat{p}),\hat{p}) = \delta_0+\mu, & k = k_1(\hat{p}), \\ = \hat{r}(\hat{p}) = \delta_0+\mu, & \forall k \in [k_1(\hat{p}), k_2(\hat{p})], \\ = g_k(k_2(\hat{p}),\hat{p}) = \delta_0+\mu, & k = k_2(\hat{p}), \\ < \delta_0+\mu, & \forall k \in (k_2(\hat{p}), k_g(\hat{p},n)), \end{cases}$$

where $\hat{p} \equiv \{p: \ \hat{r}(p) = \delta_0+\mu\}$. Therefore, $\forall k \in [k_1(\hat{p}), k_2(\hat{p})]$, we see that $g_k(k,\hat{p}) = \tilde{\delta}(k,\hat{p},n)+\mu = \delta_0+\mu$, that is, there are a continuum of possible long-run equilibria with diversified production in $\forall k \in (k_1(\hat{p}), k_2(\hat{p}))$ and two possible equilibria specified to either good in $k = k_1(\hat{p})$ or $k_2(\hat{p})$. Finally, from Eq. (1c′), function $\hat{r}(p)$ is a monotonic function which satisfies $\sup \hat{r}(p) = \infty$ and $\inf \hat{r}(p) = 0$. Therefore, $\hat{p}$, defined as mentioned above, is uniquely determined. □

B.2. Proof of Proposition 1.1b

From the analysis of Section 2, when $\delta_c+\delta_C=0$, $\forall p \in (0,\hat{p})$ (resp. $\forall p \in (\hat{p},\infty)$), this economy is specialized in good 2 (resp. good 1) and $g_{kk}(k,p) < 0$ in the steady state. Therefore, k satisfies that $g_k(k,p) = \delta_0+\mu$ is unique $\forall p \neq \hat{p}$, because $g_k(k,p)$ is continuous and monotonic decreasing function with respect to k in the steady state. □

B.3. Proof of Proposition 1.2

By the analysis above, $\lim_{k\to 0} \tilde{\delta}(k,p,n)+\mu = \delta_0+\mu < \lim_{k\to 0} g_k(k,p) = \infty$ and $\lim_{k\to k_g(p,n)} \tilde{\delta}(k,p,n)+\mu > \lim_{k\to k_g(p,n)} g_k(k,p) = n+\mu$. Moreover, because $\tilde{\delta}(k,p,n)+\mu$ is a strictly monotonic increasing function and $g_k(k,p)$ is a monotonic decreasing function with respect to k, there is unique steady state equilibrium in the region in which $k \in (0, k_g(p,n))$ necessarily. $\square$

B.4. Proof of Proposition 1.3a

First, if $\delta_c + \delta_C < 0$, $\partial(\tilde{g}(k,p,n)+\mu)/\partial k < 0$, $\forall k \in [0, k_g(p,n))$. On the other side, $\forall p \in \Omega$ and $\forall k \in (k_1(p), k_2(p)) \subset [0, k_g(p,n))$, $g_{kk}(k,p) = 0$. Therefore, if $\tilde{g}(k,p,n)+\mu = g_k(k,p)$ at $\bar{k}_d \in (k_1(p), k_2(p))$, $\tilde{g}(k_1(p), p, n)+\mu > g_k(k_1(p), p)$ and $\tilde{g}(k_2(p), p, n)+\mu < g_k(k_2(p), p)$. Second, $\lim_{k\to 0} \tilde{\delta}(k,p,n)+\mu = \delta_0 + \mu < \lim_{k\to 0} g_k(k,p) = \infty$ and Assumption 10 there is unique steady state equilibrium $\bar{k}_1$ in the region in which $k \in (0, k_1(p))$ necessarily. Finally, $\lim_{k\to k_g(p,n)} \tilde{\delta}(k,p,n)+\mu > \lim_{k\to k_g(p,n)} g_k(k,p) = n+\mu$ and Assumption 10 there is unique steady state equilibrium $\bar{k}_2$ in the region in which $k \in (k_2(p), \infty)$ necessarily. Therefore, there are three possible equilibria $\bar{k}_d \in (k_1(p), k_2(p))$, $\bar{k}_1 \in (0, k_1(p))$, and $\bar{k}_2 \in (k_2(p), \infty)$, $\forall p \in \Omega$. $\square$

B.5. Proof of Proposition 1.3b

We can distinguish following two cases: (i) $p = P_2$ and (ii) $p = P_1$. We will provide only a proof for the case (i) here, and the same basic argument is valid for the other cases. First, if $p = P_2$, $\tilde{g}(k_2(P_2), P_2, n) + \mu = g_k(k_2(P_2), P_2, n)$ because of definitions of P_2. Moreover, using property $\lim_{k\to k_g(p,n)} \tilde{\delta}(k,p,n)+\mu > \lim_{k\to k_g(p,n)} g_k(k,p) = n+\mu$ and Assumption 10, there is unique steady state equilibrium, say $\bar{k}_2 = k_2(P_2)$, in the region in which $k \in [k_2(P_2), \infty)$ necessarily. Second, in the region $k \in (k_1(P_2), k_2(P_2))$, $\tilde{g}(k, P_2, n) + \mu > g_k(k, P_2)$, because $\partial(\tilde{g}(k, P_2, n)+\mu)/\partial k < 0$ and $g_{kk}(k, P_2) = 0$. Finally, from the properties $\lim_{k\to 0} \tilde{\delta}(k, P_2, n)+\mu = \delta_0 + \mu < \lim_{k\to 0} g_k(k, P_2) = \infty$, $\tilde{g}(k_1(P_2), P_2, n)+\mu > g_k(k_1(P_2), P_2)$, and Assumption 10, there is unique steady state equilibrium, say $\bar{k}_1$, in the region $k \in (0, k_1(P_2)]$ necessarily. Then we have there are two possible equilibria $\bar{k}_1 \in (0, k_1(P_2)]$ and $\bar{k}_2 = k_2(P_2) \in [k_2(P_2), \infty)$ at $p = P_2$. $\square$

B.6. Proof of Proposition 1.3c

The above analyses (i.e., the proofs of Proposition 1.1 to 1.3b) indicate that if $\forall p \notin \Omega \cup \{P_1\} \cup \{P_2\}$, $\bar{k} \notin [k_1(p), k_2(p)]$. Therefore, we can distinguish following two cases: (i) $\tilde{\delta}(k,p,n)+\mu < g_k(k,p)$ and (ii) $\tilde{\delta}(k,p,n)+\mu > g_k(k,p)$, $\forall k \in [k_1(p), k_2(p)]$. Moreover, using properties

$\lim_{k \to 0} \tilde{\delta}(k,p,n) + \mu = \delta_0 + \mu < \lim_{k \to 0} g_k(k,p) = \infty$, $\lim_{k \to k_g} \tilde{\delta}(k,p,n) + \mu > \lim_{k \to k_g} g_k(k,p)$, and Assumption 10, there is unique long-run equilibrium $\bar{k} \in (k_2(p), \infty)$ in case (i), and $\bar{k} \in (0, k_1(p))$ in case (ii). □

Appendix C. Proof of Proposition 2

C.1. Proof of Proposition 2.1

The model's behavior can be analyzed by straightforward application of the Routh Theorem. Since $\bar{c} > 0$ and $\sigma > 0$, we know that if $pg_{kk} - (\delta - n)(\delta_c + \delta_C) < 0$, that is, $\xi_2 < 0$ and $\xi_3 > 0$, this steady state is locally saddle-point stable. □

C.2. Proof of Proposition 2.2

Since $\bar{c} > 0$, $\sigma > 0$, and $pg_{kk} - (\delta - n)(\delta_c + \delta_C) > 0$, we know that $\xi_2 \xi_3 > 0$. Therefore, we consider two cases: (i) $0 < \xi_2 < \xi_3$ which means unstable equilibrium, and (ii) $\xi_2 < \xi_3 < 0$ which means indeterminate equilibrium. In case (i) we can derive $(\delta - n) + [u_c \delta_C / (v_{cc} + v_{cC})] > 0 \Leftrightarrow \delta_C < -(\delta - n)(v_{cc} + v_{cC}) / u_c$ and in case (ii) we can observe $(\delta - n) + [u_c \delta_C / (v_{cc} + v_{cC})] < 0 \Leftrightarrow \delta_C > -(\delta - n)(v_{cc} + v_{cC}) / u_c > 0$. Therefore, we obtain Proposition 2.2. □

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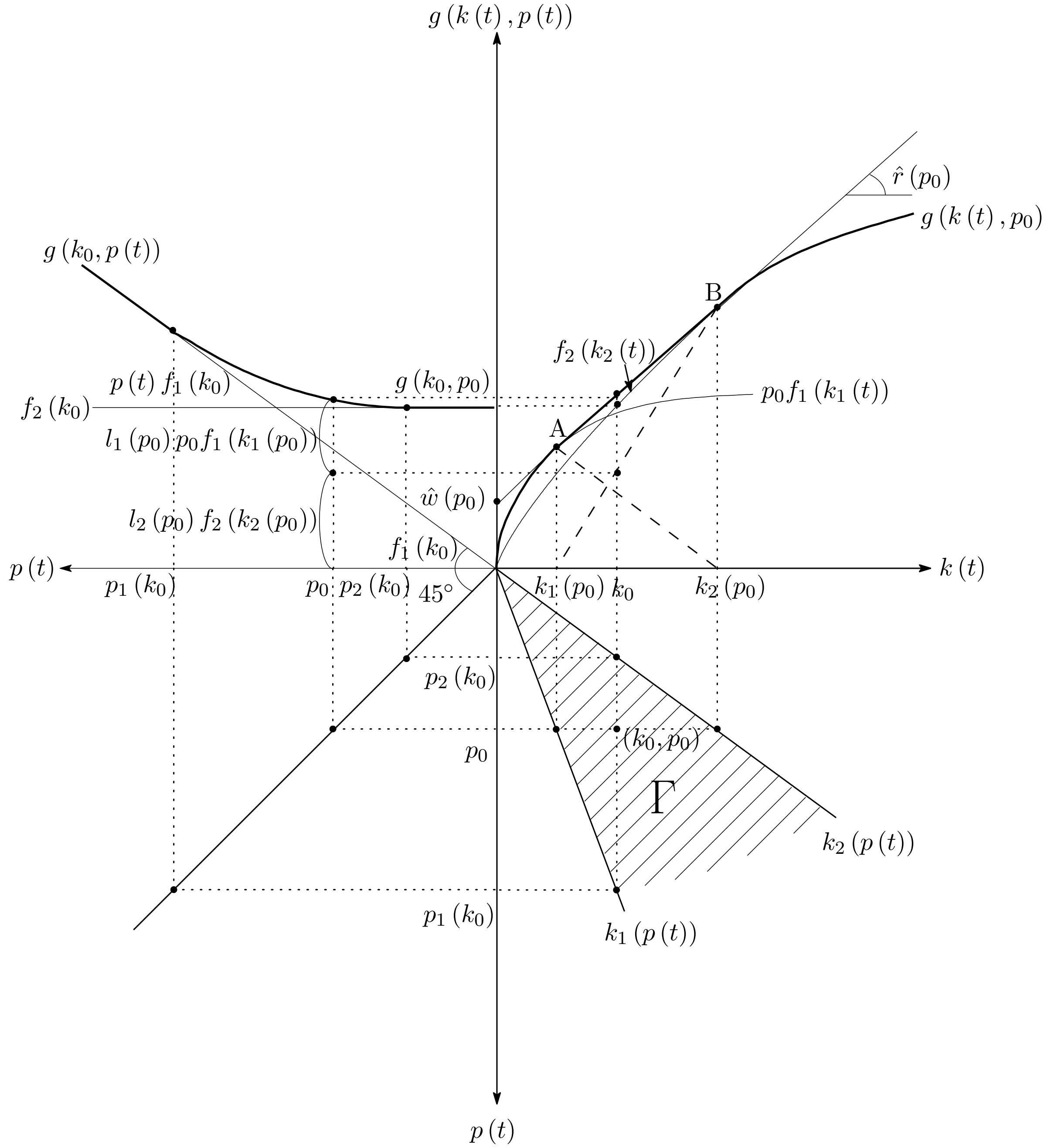

Figure 1: GDP function $g\left(k\left(t\right),p\left(t\right)\right)$

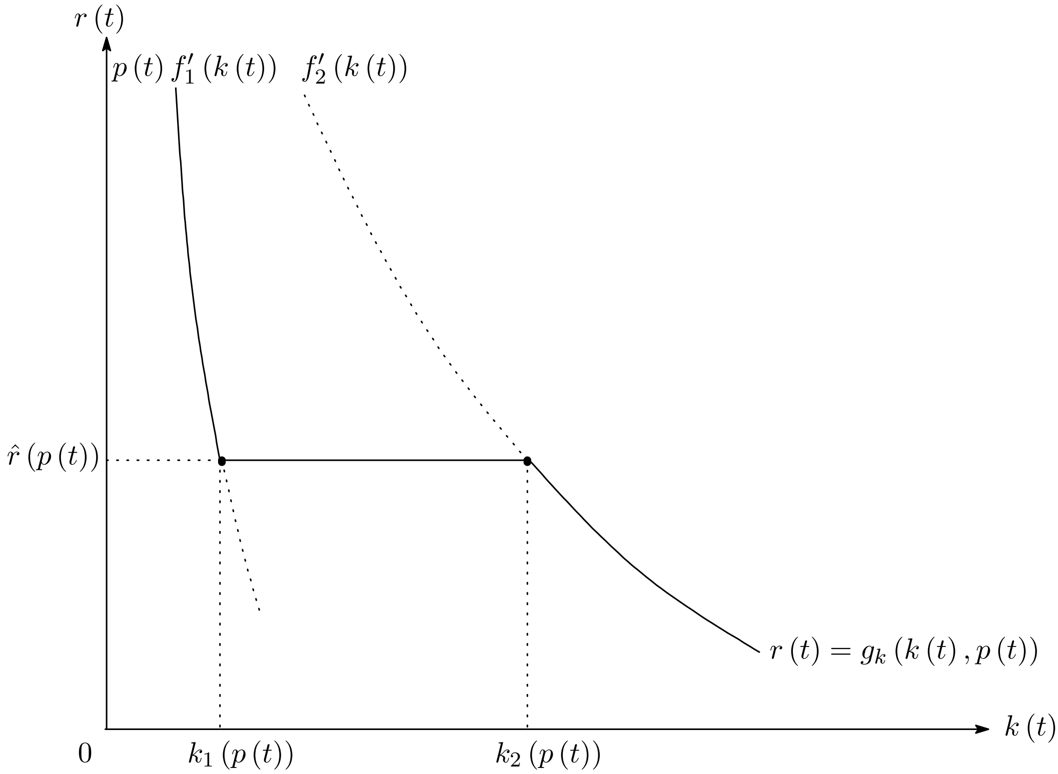

Figure 2: Demand function for capital

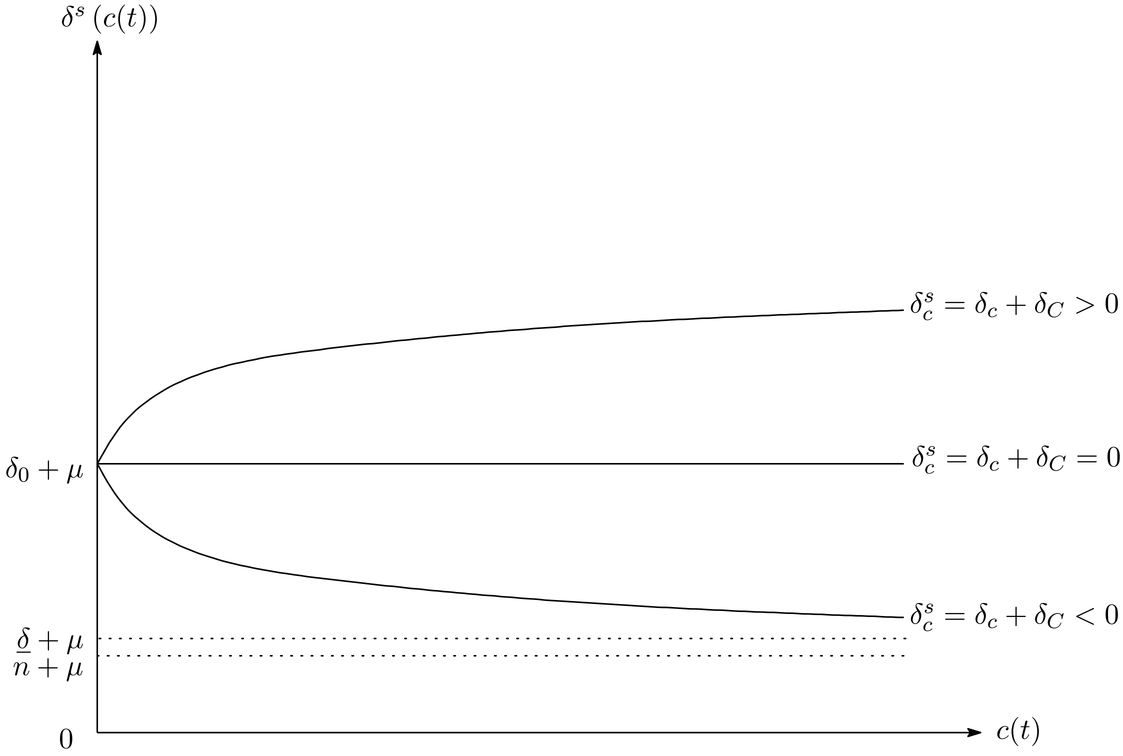

Figure 3: The social marginal impatience

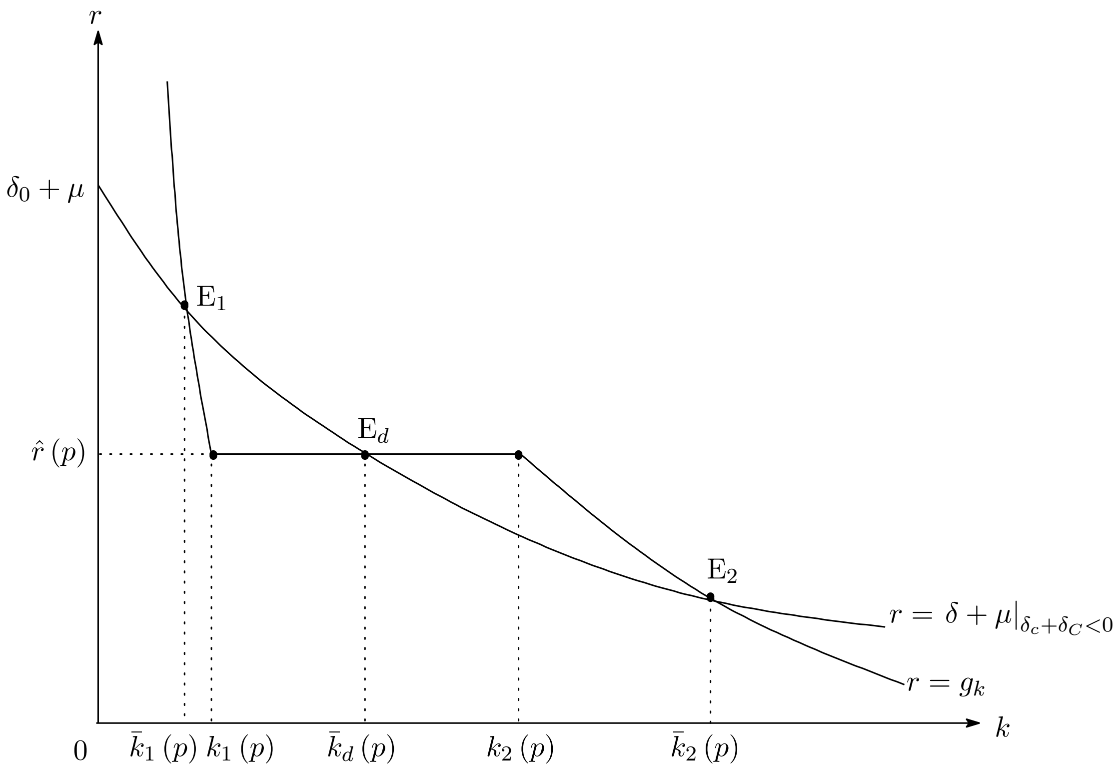

Figure 4: Multiple long-run equilibria in capital stock ($\delta_c + \delta_C < 0$)

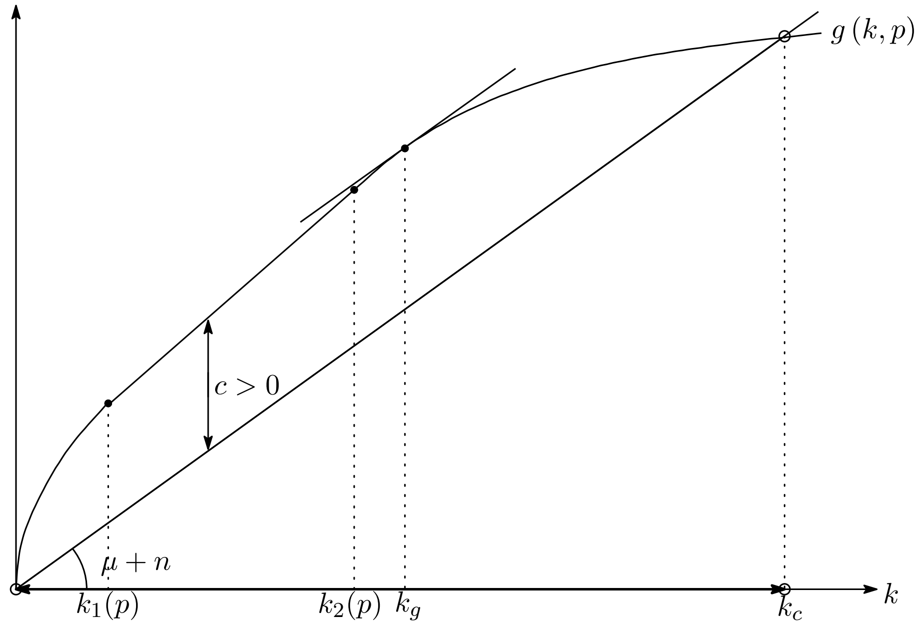

Figure 5: $k_c(p,n)$ and $k_g(p,n)$